SUMMABILITY OF FOURIER SERIES AT GENERALIZED LEBESGUE POINTS

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Abstract. A one-parameter class of generalized Lebesgue points is considered. The means of Fourier series, which are generated by linear semi-continuous summation methods, are introduced. In the case of quasi-convex summing sequences, the convergence of means at each generalized Lebesgue point is established. Estimates of deviations of means from their generating function are proposed.

Keywords: quasi-convex summation methods, deviation estimates, summability almost everywhere

1. Points of summability. Let $L(Q)$ be the class of arbitrary 2π-periodic functions $f(x)$, which are summable on $[-\pi,\pi]$,

$$c_k(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\exp(-ikt)\,dt\,,\qquad k=0,\pm1,\pm2,...$$

are Fourier coefficients of any such function and

$$\Lambda=\{\lambda_k(h);\,h>0,\,k=0,1,...;\ \lambda_0(h)=1\} \tag{1}$$

is an infinite, generally speaking, arbitrary sequence determined by the parameter $h>0$ values. We study the behavior at $h\to+0$ of the families of linear means of the Fourier series

$$U_h(f)=U(f,x;\lambda,h)=\sum_{k=-\infty}^{\infty}\lambda_{|k|}(h)c_k(f)\exp(ikx) \tag{2}$$

at the points at which

$$\lim_{k\to\infty} R_k(f,x;\eta,\gamma)=0\,. \tag{3}$$

Here

$$R_k(f,x;\eta,\gamma)=\sup_{j=0,1,...;\ j<\log_2(2\pi k)}\frac{k}{2^j\eta_\gamma(j)}\int_{-2^j/k}^{2^j/k}|\,f(x+t)-f(x)\,|\,dt,\quad k=1,2,...,$$

$\{\eta_\gamma\}$ is a family of positive functions $\eta_\gamma=\eta_\gamma(\tau)$, determined by the values of parameter $\gamma>0$ and increasing in the arguments $\tau\ge0$ and γ; at the same time we assume that $\eta_\gamma(0)=1$ and the series

$$\sum_{j=0}^{\infty}\frac{1}{\eta_\gamma(j)} \tag{4}$$

is convergent.

Lemma. The relation (3) holds almost everywhere in Q for any $f\in L(Q)$.

The assertion of the lemma (and its analogue for functions of two variables) in the case of $\eta_\gamma(j)=2^{\gamma j}$, $\gamma>0$, was established in [1]. The points with property (3) are called as generalized Lebesgue points.

2. Estimates of dodge. We will consider the quasi-convex sequences (1), that is, those for which the sum

$$\sum(h,\lambda)=\max_k|\lambda_k(h)|+\sum_{k=0}^{\infty}(k+1)|\lambda_k(h)-2\lambda_{k+1}(h)+\lambda_{k+2}(h)|$$

uniformly on h is bounded. In particular, convex (concave) and piecewise convex sequences possess the property of quasi-convexity.

Theorem 1. Let the family of functions, $\eta_\gamma=\eta_\gamma(\tau)$, $\gamma>0$, be such that the series (4) and

$$\sum_{j=0}^{\infty}\frac{\eta_\gamma(j)}{2^j} \qquad (5)$$

are convergent. Let also the sequence (1) be quasi-convex and

$$\sup_{N=0,1,...}\{|\lambda_N(h)|\cdot\eta_\gamma\left(\log_2 2\pi(N+1)\right)\cdot\log_2 2\pi(N+1)\}<\infty \quad (h>0). \qquad (6)$$

Then the series (2) is convergent for all $h>0$ at each generalized Lebesgue point, and the estimate

$$|\sum_{k=-\infty}^{\infty}\lambda_{|k|}(h)c_k(f)\exp(ikx)-f(x)|\le C_{\eta,\gamma}\sum_{k=0}^{\infty}(k+1)|\Delta^2\lambda_k(h)|\,R_{k+1}(f,x;\eta,\gamma)$$

holds.

Here and hereinafter, C denotes constants, generally speaking, different and depending only on explicitly indicated indices.

As examples of functions η_γ satisfying the convergence conditions for series (4), (5), we indicate the following:

$$\eta_\gamma(\tau)=2^{\gamma\tau},\ \tau\ge 0,\ 0<\gamma<1; \qquad (7)$$

$$\eta_\gamma(\tau)=(\tau+1)^{\gamma+1},\ \tau\ge 0,\ \gamma>0. \qquad (8)$$

3. Summability almost everywhere.

Theorem 2. Let the conditions of Theorem 1 are valid and

$$\lim_{h\to+0}\lambda_k(h)=1, \qquad k=0,1,...$$

Then the relation

$$\lim_{h\to+0}U(f,x;\lambda,h)=f(x)$$

holds at every generalized Lebesgue point, that is, almost everywhere in Q.

4. Exponential summation methods. Examples. Let now $\lambda_k(h)=\lambda(x,h)|_{x=k}$, $k=1,2,...$, where $\lambda(x,h)=\exp(-h\varphi(x))$, $\exp(-h\varphi(0))=1$, and the function $\varphi(x)$ is continuous on $[0,+\infty)$ and twice differentiable on $(0,+\infty)$. Refer to the examples.

1) Consider $\varphi(x)=\ln(x+1)$, then $\exp(-h\varphi(N))=(N+1)^{-h}$, $N=0,1,...$, and the summing sequence (1) is convex. If η_γ defined by the relation (7), then it easy to see what the results of [1] are not applicable to this summation method. The condition (6) is satisfied, however, in the case of (8) with an arbitrary fixed one $\gamma>0$. Note, that along with the fact of summability of Fourier series almost everywhere by the method

$$\lambda_k(h)=\frac{1}{(k+1)^h}, \quad k=0,1,..., \quad h\to+0$$

(see [2]) , now the character of summability points is also established.

2) In the case of $\varphi(x)=x^\alpha$, the sequence (1) is convex with $0<\alpha\le 1$ and piecewise convex with $\alpha>1$; therefore, its quasi-convexity condition is satisfied. Here the relation (6) holds when (7) occurs. Really, $\exp(-h\varphi(N))\cdot(N+1)^\beta<C_{h,\beta}$, $N=0,1,...$ with any fixed β, chosen from the condition $0<\gamma<\beta<1$. Note that a particular case of the summation method $\lambda_k(h)=\exp(-hk),\ k=0,1,...$ is the classical Poisson-Abel method.

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