ABOUT FOURIER OF REPRESENTATIONS THE SOLUTION OF THE MIXED TASK FOR THE HEAT CONDUCTIVITY EQUATION

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Abstract **–** Object of research is regional tasks and their spectral properties for the equation like heat conductivity with the deviating argument. The foundation of the general theory of the ordinary differential equations with the deviating argument was laid in Myshkis A.D. works. Myshkis A.D. researches continued by Elsgolts L.E. and Norkin S.B. in the Soviet Union. In the USA in this direction Bellman R. and Cook K.L researched. Methods of the complex analysis, theory of operators, theories of the differential equations, the spectral theory of differential operators and the theory of regular expansions are applied in work. In article influence of a deviation of argument on spectral properties of the operator of heat conductivity is studied; "spectral" decomposition of solutions of the classical mixed task for the heat conductivity equation is received.
Keywords **-** Hilbert space, heat conductivity equation, Fourier submissions, spectral properties.

I. INTRODUCTION

The theory of the differential equations with the deviating argument belongs to number of the relatively young and roughly developing sections of the theory of the ordinary differential equations. There is a number of monographs, entirely or partially devoted to various aspects of this theory. We will specify, first of all, Myshkis A.D. monographs. [1], Elsgoltsa L.E. [2; 3], Krasovsky N. N. [4], Pinni E. [5], Bellman R. and Cook K.L. [6], Norkina of Page B. [7]. The equations with the late argument appear, for example, every time when in the considered physical or technical task force operating on a material point depends on speed or the provision of this point not only at present, but also at some moment preceding this.
For the equation with the deviating argument the considerable number of mathematical works is devoted to creation of the theory of boundary tasks in recent years. Now one of the directions in this theory is developed by Azbelevy N. V. and its pupils [14].
Problem definition. Let $\Omega \subset R^2$ - the square limited to pieces:
$AB: 0 \le t \le 1,\ x=0;\quad BC: 0 \le x \le 1, t=1;\quad CD: 0 \le t \le 1, x=1;\quad DA: 0 \le x \le 1, t=0.$
Through $C^{2,1}(\Omega)-$ we will designate a set of functions $u(x,t)$ twice continuously differentiable on x and once continuously differentiable on t in area Ω. The border of area Ω is understood as set of pieces $\partial\Omega = AB \cup AD \cup CD$.
We will consider in Hilbert space $L^2(\Omega)$ the mixed task for the heat conductivity equation:

$$Tu = u_t(x,t) - u_{xx}(x,t) = f(x,t), \quad (x,t) \in \Omega. \tag{1}$$

$$u\big|_{t=0} = 0; \tag{2}$$

(3)

$$u\big|_{x=0}=0; \qquad u\big|_{x=1}=0 .$$

where $f(x,t)\in L^2(\Omega)$.
To find Fourier decomposition of the solution of the mixed task (1)-(3).
The purpose - to receive Fourier submissions of solutions of a task (1)-(3).

II. MATERIAL AND METHODS

The *regular solution* of a task (1)-(3) we will call the function $u\in C^{2,1}(\Omega)\cap C(\overline{\Omega})$ turning into identity the equation (1) and regional conditions (2)-(3).
We will call function $u\in L^2(\Omega)$ the *strong solution* of a task if there is a sequence of functions $u_n\in C^{2,1}(\Omega)\cap C(\overline{\Omega}),\, n=1,2,...$ and meeting regional statements of the problem such $\{u_n\}$ and $\{Lu_n\},\, n=1,2,...$, as meets in $L^2(\Omega)$ respectively to u and f at $n\to\infty$.
The regional task (1)-(3) is called *strongly solvable* if the strong solution of a task exists for any right part $f(x,t)\in L^2(\Omega)$ and only.[1-5]
In work methods of the complex analysis, theory of operators, theories of the differential equations, the spectral theory of differential operators and the theory of regular expansions are used.

III. RESULTS

Through S we will designate the operator determined by a formula

$$Su(x,t)=u(x,1-t).$$

In space $L^2(\Omega)$, it is obvious that S the self-conjugate and unitary operator meeting a condition $S^2=I$, where I - the single operator.
Affecting with the operator S both members of equation (1), we have

$$STu=Sf,\Rightarrow \quad u=(ST)^{-1}SF .$$

Now we investigate spectral properties of the operator ST. For this purpose we will consider a spectral task:

$$ST\upsilon=\lambda\upsilon \text{ или } T\upsilon=\lambda S\upsilon . \qquad (4)$$

We look for the solution of this task in a look:

$$\upsilon(x,t)=\omega(x)\varphi(t) .$$

Having substituted this expression in the equation (4), we will receive

$$\omega(x)\dot{\varphi}(t)-\omega''(x)\varphi(t)=\lambda\omega(x)\varphi(1-t) ,$$

$$-\omega''(x)\varphi(t)=\left[\lambda\varphi(1-t)-\dot{\varphi}(t)\right]\omega(x) ,$$

$$-\frac{\omega''(x)}{\omega(x)} = \frac{\left[\lambda\varphi(1-t) - \dot{\varphi}(t)\right]}{\varphi(t)} = \mu^2 .$$

Therefore,

$$-\omega''(x) = \mu^2 \omega(x) ,$$
$$\omega(0) = 0, \quad \omega(1) = 0 .$$

from where $\omega_n(x) = \sqrt{2}\sin n\pi x, \quad n = 1,2,...$ and $\mu_n^2 = n^2, \quad n = 1,2,...$.
For functions $\varphi(t)$ we will receive an infinite series of spectral tasks:

$$\lambda\varphi(1-t) - \dot{\varphi}(t) = n^2\varphi(t), \quad \dot{\varphi}(t) + n^2\varphi(t) = \lambda\varphi(1-t), \quad n = 1,2,...$$
$$\varphi(0) = 0 .$$

Thus, at everyone fixed values n it is necessary to solve a spectral problem:

$$\begin{cases} \dot{\varphi}(t) + n^2\varphi(t) = \lambda\varphi(1-t), \\ \varphi(0) = 0, \end{cases} \quad n = 1,2,... \tag{5}$$

We will consider more general task:

$$\begin{cases} \dot{\psi}(t) + a\psi(t) = \mu\psi(1-t), \\ \psi(0) = 0, \end{cases} \tag{6}$$

where a - any complex (generally speaking) constant, and μ spectral parameter.

Having differentiated the equation and having used a boundary condition (6), we will receive a problem of Storm-Liouville:

$$\ddot{\psi}(t) + a\dot{\psi}(t) = -\mu\psi(1-t) ,$$
$$\ddot{\psi}(t) + a\left[\mu\dot{\psi}(1-t) - a\psi(t)\right] = -\mu\left[\mu\psi(t) - a\psi(1-t)\right], \Rightarrow$$
$$\ddot{\psi}(t) + a\mu\dot{\psi}(1-t) - a^2\psi(t) = -\mu^2\psi(t) + a\mu\psi(1-t), \Rightarrow$$
$$\ddot{\psi}(t) - a^2\psi(t) = -\mu^2\psi(t), \Rightarrow$$
$$-\ddot{\psi}(t) + a^2\psi(t) = \mu^2\psi(t), \quad \psi(0) = 0, \dot{\psi}(1) + a\psi(1) = 0.$$

If function $\psi(t)$ is own function of a regional task (6), it is also own function for a problem of Storm-Liouville:

$$-\ddot{\psi}(t) + a^2\psi(t) = \mu^2\psi(t), \tag{7}$$

$$\dot{\psi}(1) + a\psi(1) = 0. \tag{8}$$

$$\psi(0) = 0. \tag{9}$$

Now we will assume the return, i.e. let function $\psi(t)$ be own function of a problem of Storm-Liouville (7)-(9), then whether there will be it own function for a task (6)?
We will find own functions of a regional task (7)-(9). From the equation (7) we have

$$-\ddot{\psi}(t)=\left(\mu^2-a^2\right)\psi(t).$$

Believing $\nu^2=\mu^2-a^2$, we will receive

$$-\ddot{\psi}(t)=\nu^2\psi(t),$$

which common decision has an appearance:

$$\psi(t)=A\cos\nu t+B\frac{\sin\nu t}{\nu},$$

where A, B - any constants. Having substituted this expression in boundary conditions (8)-(9), we have

$$\dot{\psi}(t)=-\nu A\sin\nu t+B\cos\nu t,$$
$$\dot{\psi}(1)=-\nu A\sin\nu+B\cos\nu,$$
$$-\nu A\sin\nu+B\cos\nu+a\left[A\cos\nu+B\frac{\sin\nu}{\nu}\right]=0,$$
$$\psi(0)=A=0$$

Therefore,

$$B\cos\nu+aB\frac{\sin\nu}{\nu}=\left(\cos\nu+a\frac{\sin\nu}{\nu}\right)=0.$$

As, that $B\neq 0$, then $\cos\nu+a\frac{\sin\nu}{\nu}=0$, i.e. own values of a regional task

$$-\ddot{\psi}(t)=\nu^2\psi(t),\quad \dot{\psi}(1)+a\psi(1)=0,\quad \psi(0)=0$$

are squares of roots of the equation:

$$\Delta(\nu)=\cos\nu+a\frac{\sin\nu}{\nu}=0,$$

and own functions have an appearance:

$$\psi(t)=B\frac{\sin\nu}{\nu}.$$

Having substituted this expression in the equation (7), we have

$$B\cos\nu t + Ba\frac{\sin\nu t}{\nu} = \mu B\frac{\sin\nu(1-t)}{\nu}.$$

Having reduced by B both parts of equality, we have

$$\cos\nu t + a\frac{\sin\nu t}{\nu} = \mu\frac{\sin\nu(1-t)}{\nu}.$$

We will transform the right part of this equality:

$$\mu\frac{\sin\nu(1-t)}{\nu} = \frac{\mu}{\nu}\left[\sin\nu\cos\nu t - \cos\nu\sin\nu t\right] = \mu\frac{\sin\nu}{\nu}\cos\nu t - \mu\frac{\cos\nu}{\nu}\sin\nu t =$$
$$= -\mu\frac{\cos\nu}{a}\cos\nu t - \mu\frac{\cos\nu}{\nu}\sin\nu t = -\mu\cos\nu\left[\frac{\cos\nu t}{a} + \frac{\sin\nu t}{\nu}\right] = -\frac{\mu}{a}\cos\nu\left[\cos\nu t + a\frac{\sin\nu t}{\nu}\right]$$

Therefore,

$$\cos\nu t + a\frac{\sin\nu t}{\nu} = -\frac{\mu}{a}\cos\nu\left[\cos\nu t + a\frac{\sin\nu t}{\nu}\right],$$

from where

$$-\frac{\mu}{a}\cos\nu t = 1, \quad \cos\nu = -\frac{a}{\mu}, \Rightarrow \quad -\frac{a}{\mu} + a\frac{\sin\nu}{\nu} = 0, \quad \frac{\sin\nu}{\nu} = \frac{1}{\mu}, \quad \sin\nu = \frac{\nu}{\mu}, \Rightarrow$$
$$\sin^2\nu + \cos^2\nu = 1, \Rightarrow \quad 1 + \frac{\nu^2}{\mu^2} + \frac{a^2}{\mu^2} = \frac{\nu^2 + a^2}{\mu^2}, \Rightarrow \quad \mu^2 = \nu^2 + a^2, \quad \mu = \pm\sqrt{\nu^2 + a^2},$$
$$\cos\nu = -\frac{a}{\pm\sqrt{\nu^2 + a^2}} = \mp\frac{a}{\sqrt{\nu^2 + a^2}}.$$

Thus, if function $\psi(t)$ is own function of a problem of Storm Liouville (7)-(9), it is own function of also regional task (6), where

$$\mu = \frac{a}{\cos\nu}, \quad \cos\nu + a\frac{\sin\nu}{\nu} = 0.$$

We proved the following lemma.
Lemma 1 Function $\psi(t)$ is own function of a regional task (6) in only case when it is own function of a regional problem of Storm-Liouville (7)-(9).
If a the material size, a regional task (6) is self-conjugate therefore the problem of Storm-Liouville also is self-conjugate and therefore has no the attached functions, so rated own functions of a regional task (6) form orthonormalized basis of space $L^2(\Omega)$.
Lemma 2 If a the material constant, rated own functions of a regional task (6) form orthonormalized basis of space $L^2(0,1)$.

We will designate own values of a regional task (5) through λ_{nm}, and own functions corresponding to them through $\psi_{nm}(t)$ $n,m=1,2,...$, and own functions of a regional task (4) through $\upsilon_{nm}(x,t)$, then equality takes place:

$$\upsilon_{nm}(x,t)=\omega_n(x)\psi_{nm}(t)=\sqrt{2}\sin n\pi x\,\psi_{nm}(t), \quad n,m=1,2,...$$

Lemma 3 Rated own functions of a regional task:

$$\upsilon_t(x,t)-\upsilon_{xx}(x,t)=\lambda\upsilon(x,1-t), \tag{10}$$

$$\upsilon\big|_{t=0}=0, \quad \upsilon\big|_{x=0}=0, \quad \upsilon\big|_{x=1}=0 \tag{11}$$

form orthonormalized basis of space $L^2(\Omega)$.

Proof. Orthogonality of own functions of a regional task (10)-(11) follows from symmetry of the operator ST therefore it is enough to prove completeness of system $\{\upsilon_{nm}(x,t)\}$, $n,m=1,2,...$ of own functions.

Let at some functions $f(t,x)\in L^2(\Omega)$ equality take place:

$$(f,\upsilon_{nm})=\int_0^1\int_0^1 f(x,t)\upsilon_{nm}(x,t)dxdt=0, \quad n,m=1,2,..$$

Then owing to Fubini's theorem it is had

$$\int_0^1\int_0^1 f(x,t)\upsilon_{nm}(x,t)dxdt=\int_0^1\int_0^1 f(x,t)\sqrt{2}\sin n\pi x\cdot\psi_{nm}(t)dxdt=\sqrt{2}\int_0^1\left[\int_0^1 f(x,t)\sin n\pi xdx\right]\psi_{nm}(t)dt=0$$

$$\int_0^1\left|\sqrt{2}\int_0^1 f(x,t)\sin n\pi xdx\right|^2 dt=\sum_{m=1}^{+\infty}\sqrt{2}\left|\int_0^1\left(\int_0^1 f(x,t)\sin n\pi xdx\right)\psi_{nm}(t)dt\right|^2=0$$

$$\int_0^1 f(x,t)\sin n\pi xdx=0, \quad n=1,2,..,\Rightarrow$$

$$\int_0^1|f(x,t)|^2dx=\sum_{n=1}^{\infty}\left|\int_0^1 f(x,t)\sqrt{2}\sin n\pi xdx\right|^2=0,\Rightarrow$$

$$\int_0^1|f(x,t)|^2dx=0,\Rightarrow \quad \int_0^1\int_0^1|f(x,t)|^2dxdt=0,\Rightarrow f(x,t)=0$$

almost everywhere in area Ω, as was to be shown.

In our case $a=n^2$ and $\mu=\lambda$, therefore we can formulate the following lemma.

Lemma 4 Regional task (5) has an infinite set of own values:

$$\lambda_{nm}=n^2+v_{nm}^2, \quad n,m=1,2,..$$

where ν_{nm} - roots of the equations

$$\cos\nu_{nm} + \frac{n^2 \sin\nu_{nm}}{\nu_{nm}} = 0, \quad n, m = 1, 2, ... \tag{12}$$

and corresponding to them own functions

$$\varphi_{nm}(t) = B_{nm} \frac{\sin\nu_{nm} t}{\nu_{nm}}, \quad t \in [0,1]$$

where B_{nm} - rated coefficients, and $\nu_{nm}^2 > 0$.
We will assume that $u(x,t) \in D(ST)$, then $ST \in L^2(\Omega)$ also equality takes place:

$$STu = \sum_{n,m}^{\infty} (STu, \upsilon_{nm})\upsilon_{nm} = \sum_{n,m}^{\infty} (u, ST\upsilon_{nm})\upsilon_{nm} = \sum_{n,m}^{\infty} \lambda (u, \upsilon_{nm})\upsilon_{nm},$$

where $ST\upsilon_{nm} = \lambda_{nm}\upsilon_{nm}$. Therefore,

$$\|STu\|^2 = \sum_{n,m}^{\infty} \lambda_{nm}^2 |(u, \upsilon_{nm})|^2 < +\infty .$$

Then

$$(STu, u) = \left(\sum_{nm}^{\infty} \lambda_{nm}(u, \upsilon_{nm})\upsilon_{nm}, \sum_{nm}^{\infty} (u, \upsilon_{nm})\upsilon_{nm} \right) = \sum_{nm}^{\infty} \lambda_{nm} |(u, \upsilon_{nm})|^2 \ge \sum_{nm}^{\infty} u^2 |(u, \upsilon_{nm})|^2 \ge \|u\|^2$$

i.e. the operator ST we will turn. We will find the return operator $(ST)^{-1}$.

$$(ST)^{-1} f = \sum_{nm}^{\infty} \left((ST)^{-1} f, \upsilon_{nm}\right)\upsilon_{nm} = \sum_{nm}^{\infty} (f, \upsilon_{nm})\upsilon_{nm} = \sum_{nm}^{\infty} \lambda_{nm}$$

As $\lim_{n,m\to\infty} \lambda_{nm} = +\infty$, that of the last equality follows quite the operator's continuity $(ST)^{-1} f$.
Now we will return to our initial task. The solution of our task has an appearance:

$$u(x,t) = (ST)^{-1} Sf = \sum_{nm=1}^{\infty} \frac{(Sf, \upsilon_{nm})}{\lambda_{nm}} \upsilon_{nm}(x,t),$$

where

$$\dot{\upsilon}_{nm}(x,t) - (\upsilon_{nm}(x,t))_{xx} = \lambda_{nm}\upsilon_{nm}(x, 1-t)$$

$$\upsilon_{nm}\big|_{t=0} = 0, \quad \upsilon_{nm}\big|_{x=0} = 0, \quad \upsilon_{nm}\big|_{x=1} = 0,$$

$$\upsilon_{nm}(x,t) = \sqrt{2}\sin n\pi x \cdot \frac{B_{nm}\sin v_{nm}t}{v_{nm}}, \quad n,m = 1,2,...$$

v_{nm} - roots of the equations (12), B_{nm} - normalizing coefficients.

IV. CONCLUSIONS

As a result of research the following theorem is proved.

Theorem

(a) The mixed task (1)-(3) for the equation of heat conductivity is strongly solvable in space $L^2(\Omega)$;

(b) The return operator $(\overline{T})^{-1}$ is quite continuous on this space and Voltaire;

(c) "Spectral" decomposition takes place:

$$u(x,t) = T^{-1}f = \sum_{nm=1}^{\infty} \frac{(Sf, \upsilon_{nm})}{\lambda_{nm}} \upsilon_{nm},$$

where $Sf(x,t) = f(x,1-t)$,

$$\frac{\partial}{\partial t}\upsilon_{nm}(x,t) - \frac{\partial^2}{\partial x^2}\upsilon_{nm}(x,t) = \lambda_{nm}\upsilon_{nm}(x,1-t),$$

$$\upsilon_{nm}\big|_{t=0} = 0, \quad \upsilon_{nm}\big|_{x=0} = 0, \quad \upsilon_{nm}\big|_{x=1} = 0,$$

$$\upsilon_{nm}(x,t) = \sqrt{2}\sin n\pi x \cdot \frac{B_{nm}\sin v_{nm}t}{v_{nm}}, \quad n,m = 1,2,...$$

v_{nm} - roots of the equations (12), B_{nm} - normalizing coefficients,

$\{\upsilon_{nm}(x,t)\}$ - orthonormalized basis of space $L^2(\Omega)$.

Thus, spectral properties of the indignant operator of heat conductivity are investigated; Fourier submission of solutions of the mixed task for the heat conductivity equation is brought.

Results of article are an essential contribution to development of the general spectral theory of regional tasks for the differential equations.

The received results can be applied in further researches of regional tasks to the differential equations and theories of operators.

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