GENERALIZED DIFFERENTIAL VECTOR DYNAMIC EULER-EQUATIONS IN GESS-FORM

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Abstract—*This paper deals with the generalized differential dynamic equations of Euler–Hess, which contain an arbitrary three-dimensional vector. From generalized equations it obtains classical equation of Euler–Hess in the first form and new forms of dynamic equations in the Euler–Hess form without the direction cosines. The integrable precession vector equation in the Euler–Gess form with two first independent integral is received.*

Index Terms—Euler–Poisson equations; Hess equations; equations in the Euler–Hess form; dynamic of rigid body; gyrostat; irregular precession.

INTRODUCTION

We consider the traditional problem of order decreasing (reduction) of the six scalar classical dynamical Euler–Poisson equations system in the vector form [1] – [3]:

$$\dot{\bar{g}}=\bar{g}\times\bar{\omega}-P_T[\bar{\rho}\times\bar{\gamma}], \tag{1}$$

$$\dot{\bar{\gamma}}=\bar{\gamma}\times\bar{\omega}, \tag{2}$$

formed by Euler's equation (1) and by Poisson's equation (2). In this equations $\bar{g}$ – vector of kinetic moment of a rigid body (RB); $\dot{\bar{g}}=(d\,\bar{g}/dt)$ – relative (local) t time derivative of the kinetic momentum vector $\bar{g}=S\bar{\omega}$ of the RB (S – diagonal inertia tensor); $\bar{\omega}$ – vector of angular velocity of RB; $\bar{\rho}$ – radius-vector of the mass center of the RB with the origin at the fixed point of the RB; P_T – body weight; $\bar{\gamma}$ – vertical unit vector; $\dot{\bar{\gamma}}=(d\,\bar{\gamma}/dt)$ – relative derivative.

These equations have three independent algebraic first integrals: 1) energy integral is the $C_1=(\bar{g}\cdot\bar{\omega})/2-P_T(\bar{\gamma}\cdot\bar{\rho})$; 2) area integral is the $C_2=(\bar{\gamma}\cdot\bar{g})$; 3) geometric integral is the $(\bar{\gamma}\cdot\bar{\gamma})=1$, where C_1, C_2 are arbitrary constants.

Three scalar products are used to determine the directional cosines of "vertical": $C_2=(\bar{\gamma}\cdot\bar{g})$, $f_1=(\bar{\gamma}\cdot\bar{\rho})$, $f_{æ}=(\bar{\gamma}\cdot\bar{æ})$. Where $\bar{æ}$ is the arbitrary three-dimensional vector.

By substituting the expression for the vertical unit vector in the Euler equations (1) the generalized equations of the Euler-Hess form are obtained – three scalar equations.

The Euler–Hess equations of particular (the first [3, p. 37] form) were obtained for the first time by the German mathematician V. Hess in 1890 and to date they have been considered in many papers without changes. V. Hess got them on the basis of three classical integrals of the Euler-Poisson equations, excluding from Euler equations the direction cosines – the coordinates of the $\bar\gamma$ vertical unit vector. Herewith V. Hess has applied complex "symmetrical" [1, p. 27] method for solving a system of three nonlinear algebraic equations, two of which are linear, and the third equation – non-linear, a quadratic type. The complexity of V. Hess method is due to the need to obtain and converse the partial derivatives of the matrix of the system determinant of inhomogeneous equations by arbitrary constants of the equations first integrals.

The report proposes a new essentially and most simple algebraic method for determining (initially) the direction cosines of vertical based only on two integrals of energy and area. At this stage it is previously solved a simple system of three linear inhomogeneous algebraic equations (given by the scalar product of vectors) using the identities of linear vector algebra. One of the equations contains the coordinates of an arbitrary three-dimensional $\bar{æ}$ vector, the selection of which allows receiving various new dynamic equations. The generalized dynamic equations of the Euler–Hess form were obtained by this method. In these equations one only need to replace the arbitrary $\bar{æ}$ vector.

I. Generalized Equations of the Euler–Hess

These equations are written in vector form

$$\overset{*}{\bar g} = \bar g \times \bar\omega - Đ_T\left[\bar\rho \times \bar\gamma_{æ}\right], \tag{3}$$

$$\bar\gamma_{\bar{æ}} = \left(1/\varepsilon_{æ}\right)\left(c_2\left[\bar\rho \times \bar{æ}\right] + f_1\left[\bar{æ} \times \bar g\right] + f_{æ}\left[\bar g \times \bar\rho\right]\right), \tag{4}$$

where $\varepsilon_{æ} = \left(\bar g \cdot \left[\bar\rho \times \bar{æ}\right]\right)$ is the mixed product of non-collinear vectors; c_2 – arbitrary constant of the area integral; $f_1 = (T - c_1)/P_T$, $Ò = (\bar g \cdot \bar\omega)/2$ is the kinetic energy of the RB; $ñ_1$ is the arbitrary constant energy integral; $f_{æ} = \left(\bar\gamma \cdot \bar{æ}\right)$ is the function defined by the unknown $\bar\gamma_{æ}$ vertical unit vector and arbitrary assigned vector $\bar{æ}$ with coordinates in the principal central axes of the RB inertia.

The function is determined as a result of solving a quadratic equation obtained after the substituting a unit vector (2) to the first (geometric) integral $\left(\bar\gamma_{æ} \cdot \bar\gamma_{æ}\right) = 1$.

II. Classic Equations of the Euler–Hess

In the special case for $\bar{æ} = \bar g \times \bar\rho$ the classical Euler–Hess equations (Euler – equations in Hess form [1, p. 30]) with the expansion of Hess for the $\bar\gamma$ unit vector along three vectors $\bar g, \bar\rho, \bar g \times \bar\rho$ [3, p.

37] can be obtained from (3), (4). The equations have the integrating Jacobi multiplier, but have not the first integral with arbitrary constant and they cannot be integrated in quadratures.

III. Equations in the Euler–Hess Form with Constant Arbitrary Vector

In the case of a constant vector $\bar{æ} = \bar{\sigma}$ expansion (4) takes the form $\bar{\gamma}_\sigma = (1/\varepsilon_\sigma)(c_2\bar{V} + f_1[\bar{\sigma}\times\bar{g}] + + f_\sigma[\bar{g}\times\bar{\rho}])$, where $f_\sigma = (\bar{\gamma}\cdot\bar{\sigma})$, $\bar{V} = [\bar{\rho}\times\bar{\sigma}]$ is the a constant vector.

The $\sigma_1, \sigma_2, \sigma_3$ coordinates of $\bar{\sigma}$ vector may be equal to, for example, the main (central) momentums *A*, *B*, *C* of the RB inertia (than $\bar{\sigma} = \bar{\sigma}_*$ is the *inertia vector* [12]) or to the constant coordinates of the vector of gyrostat gyrostatic moment (with flywheels or gyroscopes).

IV. Equation in the Euler–Hess Form for Gyrostat

The equations of the Euler–Hess form may be of interest in the dynamics of gyrostat and in the tasks of RB orientation control, considering that modern spacecraft (including micro satellites, drones, unmanned air- and spacecraft for special air- and space missions), driven by a flywheel and strapdown inertial systems are the gyrostats, for example [4]–[11]. The particular interest are the equations forms of the Euler-Hess form with first integrals about which "almost nothing is known" [14, p. 19]. One such equations form is obtained in [13] for known [15, p. 81–87] case of "semi regular" precession with constant modulus of the kinetic momentum vector (asymmetric gyrostat) – arbitrary constant function $C_g = (\bar{g}\cdot\bar{g}) = \text{const}$ (similar to the first Euler integral in the case of Euler [1]–[3]). This equations form is obtained from generalized Euler equation (3) and has the form:

$$\dot{\bar{g}} = \bar{g}\times\bar{\omega} + c_\rho\,\bar{g}\times\bar{\rho}\ , \tag{5}$$

where c_ρ is the constant scalar function (but not arbitrary); $\bar{g}$ is an angular momentum vector of a gyrostat, containing constant vector of the gyrostatic momentum [15, pp. 19, 81].

C_g constant function (integral of gyrostat kinetic momentum), which exists only under condition $(\bar{\gamma}_*\cdot[\bar{g}\times\bar{\rho}]) = 0$ – complanarity relations for coplanar vectors $\bar{\gamma}_*, \bar{g}, \bar{\rho}$ (which are located in one plane). The C_g integral is obtained after scalar multiplication of equation (5) on the $\bar{g}$ vector and the subsequent integration of the $(\bar{g}\cdot\dot{\bar{g}}) = 0$ function (as in the Euler case [1], [2] at $\bar{\rho} = 0$). $\bar{\gamma}_*$ – unit vector is defined under condition $(\bar{\gamma}_*\cdot[\bar{g}\times\bar{\rho}]) = 0$ by expansion $\bar{\gamma}_* = a\bar{g} + b\bar{\rho}$, where *a*, *b* is the constant [15, p. 87] functions depending on the constant scalar products $C_{\rho g} = (\bar{\rho}\cdot\bar{g}) =$ const, $c_{\gamma\rho} = (\bar{\gamma}_*\cdot\bar{\rho})$ = const. This expansion follows from the generalized expansion (4) at $\bar{æ} = [\bar{g}\times\bar{\rho}]$, and at $f_æ = 0$.

Arbitrary function $C_{\rho g}$ – is constant function. It is obtained after scalar multiplication of equation (5) (or (3)) on the $\bar{\rho}$ vector and after the subsequent integration of the $(\bar{\rho}\cdot\dot{\bar{g}})=0$ function provided $(\bar{\rho}\cdot[\bar{g}\times\bar{\omega}])=0$ – complanarity of $\bar{\rho}, \bar{g}, \bar{\omega}$ vectors.

Condition $c_{\gamma\rho}=(\bar{\gamma}_*\cdot\bar{\rho})=$ const is provided, in its turn by the presence of the third constant function of RB: $2T=C_{\grave{O}}=(\bar{g}\cdot\bar{\omega})=\text{const}$. This constant function is determined after scalar multiplication of equation (5) on the $\bar{\omega}$ vector and integration of the $(\bar{\omega}\cdot\dot{\bar{g}})=0$ function also under condition of complanarity of $\bar{\rho}, \bar{g}, \bar{\omega}$ vectors. Then the $c_{\gamma\rho}=(\bar{\gamma}_*\cdot\bar{\rho})$ function is constant into force of the classical energy integral of the (1), (2) equations for gyrostat.

V. Definition of the Mass Center Vector of the Rigid Body

We consider non-traditional task of RB dynamics – the task of determining the constant $\bar{\rho}$ vector, which enforces the conditions of vectors complanarity. As an example, we solve the system of three algebraic equations defined by a system of three scalar products $(\bar{\rho}_1\cdot\bar{\sigma}_*)=c_*$ (c_* – constant, $\bar{\sigma}_*$– inertia vector, see p. 4); $(\bar{\rho}_1\cdot\bar{g})=C_{\rho g}$; $(\bar{\rho}_1\cdot[\bar{g}\times\bar{\omega}])=0$. As a result the $\bar{\rho}_1$ vector located in the plane of the $\bar{g},\bar{\omega}$ vectors is determined. Similarly, we solve the system of scalar equations $(\bar{\rho}_2\cdot\bar{\sigma}_*)=c_*$, $(\bar{\rho}_2\cdot\bar{g})=C_{\rho g}$, $(\bar{\rho}_2\cdot[\bar{\gamma}\times\bar{g}])=0$. After substituting the expressions obtained for the $\bar{\rho}$ vector in the Euler equation (3) it is obtained the new equations of the Euler–Hess form and of the Euler– Poisson with arbitrary constant functions and complanarity relations, containing the $\bar{\sigma}_*$ vector instead of a $\bar{\rho}$ vector. Complanarity relations can be used for reduction of order of three differential equations systems obtained [15, p. 136] of the Euler–Hess form.

VI. Integrable Precession Equation in the Euler–Gess Form

There is also a possibility to derive the inferable equations of the Euler-Hess form (not containing vertical unit vector) with use only the one first integral - an area integral, instead of all three classical integrals used for deriving of the generalized equations (3), (4). Such possibility appears under condition of $(\bar{\gamma}\cdot[\bar{\omega}\times\bar{\sigma}_*])=0$ – a complanarity of vectors $\bar{\gamma}$, $\bar{\omega}$, $\bar{\sigma}_*$. Provided that the Poisson equation (2) supposes a special additional first ("precession" [3, p. 242]) algebraic integral $C_\sigma=(\bar{\sigma}_*\cdot\bar{\gamma})$. This integral is as a result of scalar multiplication of the Poisson equation (2) on a vector of inertia $\bar{\sigma}_*$ and the subsequent first integration of scalar function $(\bar{\sigma}_*\cdot\dot{\bar{\gamma}})=0$.

Then the vertical unit vector $\bar{\gamma}$ is uniquely determined as a result of a solution of system of three linear inhomogeneous algebraic equations, set by three scalar products:

$$(\bar{\gamma}\cdot\bar{g})=C_2,\ \ (\bar{\gamma}\cdot\bar{\sigma}_*)=C_\sigma,\ \ (\bar{\gamma}\cdot[\bar{\omega}\times\bar{\sigma}_*])=0, \tag{6}$$

and writes in the form of

$$\bar{\gamma}_\sigma=\alpha\bar{\omega}-\beta\bar{\sigma}_*, \tag{7}$$

where: $\alpha=(\bar{w}\cdot\bar{\sigma}_*)/\varepsilon_\sigma$; $\beta=(\bar{w}\cdot\bar{\omega})/\varepsilon_\sigma$; $\bar{w}=C_2\bar{\sigma}_*-C_\sigma\bar{g}$; $\varepsilon_\sigma=(\bar{g}\cdot\bar{\omega})\cdot(\bar{\sigma}_*\cdot\bar{\sigma}_*)-(\bar{g}\cdot\bar{\sigma}_*)\cdot(\bar{\omega}\cdot\bar{\sigma}_*)\neq 0$

Substitution of a unit vector (7) to the Euler equation (1) shall transform it to the “reduced” precession vector equation of Euler–Gess form (not containing a vertical unit vector):

$$\dot{\bar{g}}=\bar{g}\times\bar{\omega}-P_T\alpha\bar{\rho}\times\bar{\omega}+P_T\beta\bar{\rho}\times\bar{\sigma}_*. \tag{8}$$

Under condition of a complanarity $(\bar{\gamma}\cdot[\bar{\omega}\times\bar{\sigma}_*])=0$ equation (8) is considered as integralable. It has the two first integrals, received of a classical integral of total energy of RB and of a geometrical integral after replacement in them of a $\bar{\gamma}$ vertical unit vector by expression (7).

The equation (8) turns out also from the generalized equation (3) at $\bar{æ}=\bar{\omega}\times\bar{\sigma}_*$ and at replacement of function f_1 by function C_σ according to a equations system (6). Thus function $f_{æ}=(\bar{æ}\cdot\bar{\gamma})=0$.

VII. Conclusions

The use of generalized Euler–Gess equations enables to obtain new equation in the Euler–Gess form with the first algebraic integrals corresponding conditions of the vectors complanarity.

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